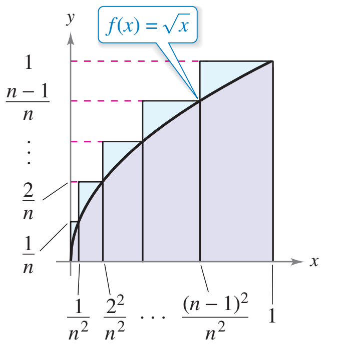
4.3 Riemann Sums and Definite Integrals

- **Understand the definition of a Riemann sum.**
- **Evaluate a definite integral using limits.**
- **Evaluate a definite integral using properties of definite integrals.**

Riemann Sums

In the definition of area given in Section 4.2, the partitions have subintervals of *equal width*. This was done only for computational convenience. The next example shows that it is not necessary to have subintervals of equal width.

EXAMPLE 1 A Partition with Subintervals of Unequal Widths



The subintervals do not have equal widths.
Figure 4.18

Consider the region bounded by the graph of

$$f(x) = \sqrt{x}$$

and the x-axis for $0 \le x \le 1$, as shown in Figure 4.18. Evaluate the limit

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta x_i$$

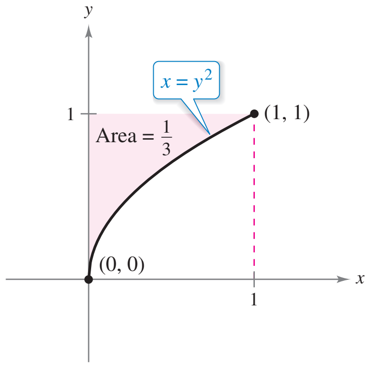
where c_i is the right endpoint of the partition given by $c_i = i^2/n^2$ and Δx_i is the width of the ith interval.

Solution The width of the ith interval is

$$\begin{aligned}\Delta x_i &= \frac{i^2}{n^2} - \frac{(i-1)^2}{n^2}\\ &= \frac{i^2 - i^2 + 2i - 1}{n^2}\\ &= \frac{2i-1}{n^2}.\end{aligned}$$

So, the limit is

$$\begin{aligned}\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta x_i &= \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{\frac{i^2}{n^2}}\left(\frac{2i-1}{n^2}\right)\\ &= \lim_{n\to\infty} \frac{1}{n^3}\sum_{i=1}^{n}(2i^2 - i)\\ &= \lim_{n\to\infty} \frac{1}{n^3}\left[2\left(\frac{n(n+1)(2n+1)}{6}\right) - \frac{n(n+1)}{2}\right]\\ &= \lim_{n\to\infty} \frac{4n^3 + 3n^2 - n}{6n^3}\\ &= \frac{2}{3}.\end{aligned}$$



The area of the region bounded by the graph of $x = y^2$ and the y-axis for $0 \le y \le 1$ is $\frac{1}{3}$.
Figure 4.19

From Example 7 in Section 4.2, you know that the region shown in Figure 4.19 has an area of $\frac{1}{3}$. Because the square bounded by $0 \le x \le 1$ and $0 \le y \le 1$ has an area of 1, you can conclude that the area of the region shown in Figure 4.18 has an area of $\frac{2}{3}$. This agrees with the limit found in Example 1, even though that example used a partition having subintervals of unequal widths. The reason this particular partition gave the proper area is that as n increases, the *width of the largest subinterval approaches zero.* This is a key feature of the development of definite integrals.

GEORG FRIEDRICH BERNHARD RIEMANN (1826-1866)

German mathematician Riemann did his most famous work in the areas of non-Euclidean geometry, differential equations, and number theory. It was Riemann's results in physics and mathematics that formed the structure on which Einstein's General Theory of Relativity is based.

See LarsonCalculus.com to read more of this biography.

In Section 4.2, the limit of a sum was used to define the area of a region in the plane. Finding area by this means is only one of *many* applications involving the limit of a sum. A similar approach can be used to determine quantities as diverse as arc lengths, average values, centroids, volumes, work, and surface areas. The next definition is named after Georg Friedrich Bernhard Riemann. Although the definite integral had been defined and used long before Riemann's time, he generalized the concept to cover a broader category of functions.

In the definition of a Riemann sum below, note that the function f has no restrictions other than being defined on the interval $[a, b]$. (In Section 4.2, the function f was assumed to be continuous and nonnegative because you were finding the area under a curve.)

Definition of Riemann Sum

Let f be defined on the closed interval $[a, b]$, and let Δ be a partition of $[a, b]$ given by

$$a = x_0 < x_1 < x_2 < \ . \ . \ . \ < x_{n-1} < x_n = b$$

where Δx_i is the width of the ith subinterval

$$[x_{i-1}, x_i].$$ ith subinterval

If c_i is *any* point in the ith subinterval, then the sum

$$\sum_{i=1}^{n} f(c_i)\,\Delta x_i, \quad x_{i-1} \le c_i \le x_i$$

is called a **Riemann sum** of f for the partition Δ. (The sums in Section 4.2 are examples of Riemann sums, but there are more general Riemann sums than those covered there.)

The width of the largest subinterval of a partition Δ is the **norm** of the partition and is denoted by $\|\Delta\|$. If every subinterval is of equal width, then the partition is **regular** and the norm is denoted by

$$\|\Delta\| = \Delta x = \frac{b-a}{n}.$$ Regular partition

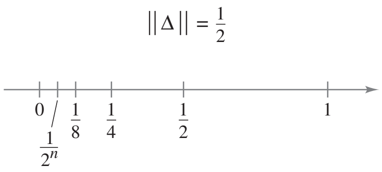
For a general partition, the norm is related to the number of subintervals of $[a, b]$ in the following way.

$$\frac{b-a}{\|\Delta\|} \le n$$ General partition

So, the number of subintervals in a partition approaches infinity as the norm of the partition approaches 0. That is, $\|\Delta\| \to 0$ implies that $n \to \infty$.

The converse of this statement is not true. For example, let Δ_n be the partition of the interval $[0, 1]$ given by

$$0 < \frac{1}{2^n} < \frac{1}{2^{n-1}} < \cdot \cdot \cdot < \frac{1}{8} < \frac{1}{4} < \frac{1}{2} < 1.$$



$n \to \infty$ does not imply that $\|\Delta\| \to 0$.

Figure 4.20

As shown in Figure 4.20, for any positive value of n, the norm of the partition Δ_n is $\frac{1}{2}$. So, letting n approach infinity does not force $\|\Delta\|$ to approach 0. In a regular partition, however, the statements

$$\|\Delta\| \to 0 \quad \text{and} \quad n \to \infty$$

are equivalent.

INTERFOTO/Alamy

Definite Integrals

FOR FURTHER INFORMATION For insight into the history of the definite integral, see the article "The Evolution of Integration" by A. Shenitzer and J. Steprāns in *The American Mathematical Monthly.* To view this article, go to *MathArticles.com.*

To define the definite integral, consider the limit

$$\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(c_i)\,\Delta x_i = L.$$

To say that this limit exists means there exists a real number L such that for each $\varepsilon > 0$, there exists a $\delta > 0$ such that for every partition with $\|\Delta\| < \delta$, it follows that

$$\left| L - \sum_{i=1}^{n} f(c_i)\,\Delta x_i \right| < \varepsilon$$

regardless of the choice of c_i in the ith subinterval of each partition Δ.

Definition of Definite Integral

If f is defined on the closed interval $[a, b]$ and the limit of Riemann sums over partitions Δ

$$\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(c_i)\,\Delta x_i$$

exists (as described above), then f is said to be **integrable** on $[a, b]$ and the limit is denoted by

$$\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(c_i)\,\Delta x_i = \int_a^b f(x)\,dx.$$

The limit is called the **definite integral** of f from a to b. The number a is the **lower limit** of integration, and the number b is the **upper limit** of integration.

REMARK Later in this chapter, you will learn convenient methods for calculating $\int_a^b f(x)\,dx$ for continuous functions. For now, you must use the limit definition.

It is not a coincidence that the notation for definite integrals is similar to that used for indefinite integrals. You will see why in the next section when the Fundamental Theorem of Calculus is introduced. For now, it is important to see that definite integrals and indefinite integrals are different concepts. A definite integral is a *number*, whereas an indefinite integral is a *family of functions.*

Though Riemann sums were defined for functions with very few restrictions, a sufficient condition for a function f to be integrable on $[a, b]$ is that it is continuous on $[a, b]$. A proof of this theorem is beyond the scope of this text.

THEOREM 4.4 Continuity Implies Integrability

If a function f is continuous on the closed interval $[a, b]$, then f is integrable on $[a, b]$. That is, $\int_a^b f(x)\,dx$ exists.

Exploration

The Converse of Theorem 4.4 Is the converse of Theorem 4.4 true? That is, when a function is integrable, does it have to be continuous? Explain your reasoning and give examples.

Describe the relationships among continuity, differentiability, and integrability. Which is the strongest condition? Which is the weakest? Which conditions imply other conditions?

EXAMPLE 2 Evaluating a Definite Integral as a Limit

Evaluate the definite integral $\int_{-2}^{1} 2x\,dx$.

Solution The function $f(x) = 2x$ is integrable on the interval $[-2, 1]$ because it is continuous on $[-2, 1]$. Moreover, the definition of integrability implies that any partition whose norm approaches 0 can be used to determine the limit. For computational convenience, define Δ by subdividing $[-2, 1]$ into n subintervals of equal width

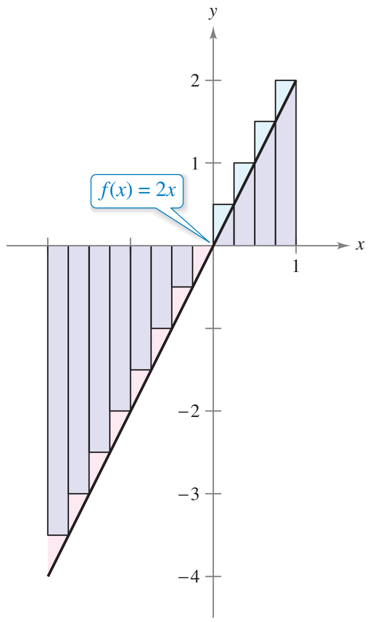
$$\Delta x_i = \Delta x = \frac{b-a}{n} = \frac{3}{n}.$$

Choosing c_i as the right endpoint of each subinterval produces

$$c_i = a + i(\Delta x) = -2 + \frac{3i}{n}.$$

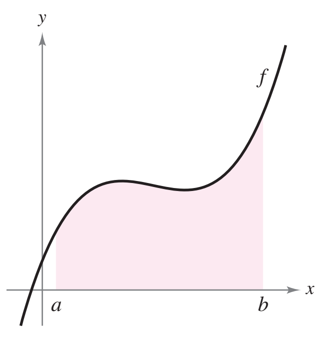
So, the definite integral is

$$\begin{aligned}
\int_{-2}^{1} 2x\,dx &= \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(c_i)\,\Delta x_i \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta x \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} 2\left(-2 + \frac{3i}{n}\right)\left(\frac{3}{n}\right) \\
&= \lim_{n\to\infty} \frac{6}{n} \sum_{i=1}^{n} \left(-2 + \frac{3i}{n}\right) \\
&= \lim_{n\to\infty} \frac{6}{n} \left(-2\sum_{i=1}^{n} 1 + \frac{3}{n}\sum_{i=1}^{n} i\right) \\
&= \lim_{n\to\infty} \frac{6}{n} \left\{-2n + \frac{3}{n}\left[\frac{n(n+1)}{2}\right]\right\} \\
&= \lim_{n\to\infty} \left(-12 + 9 + \frac{9}{n}\right) \\
&= -3.
\end{aligned}$$



Because the definite integral is negative, it does not represent the area of the region.
Figure 4.21

Because the definite integral in Example 2 is negative, it *does not* represent the area of the region shown in Figure 4.21. Definite integrals can be positive, negative, or zero. For a definite integral to be interpreted as an area (as defined in Section 4.2), the function f must be continuous and nonnegative on $[a, b]$, as stated in the next theorem. The proof of this theorem is straightforward—you simply use the definition of area given in Section 4.2, because it is a Riemann sum.



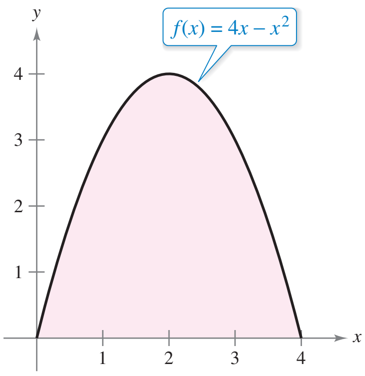
You can use a definite integral to find the area of the region bounded by the graph of f, the x-axis, $x = a$, and $x = b$.
Figure 4.22

THEOREM 4.5 The Definite Integral as the Area of a Region

If f is continuous and nonnegative on the closed interval $[a, b]$, then the area of the region bounded by the graph of f, the x-axis, and the vertical lines $x = a$ and $x = b$ is

$$\text{Area} = \int_a^b f(x)\,dx.$$

(See Figure 4.22.)



$$\text{Area} = \int_0^4 (4x - x^2)\,dx$$

Figure 4.23

As an example of Theorem 4.5, consider the region bounded by the graph of

$$f(x) = 4x - x^2$$

and the x-axis, as shown in Figure 4.23. Because f is continuous and nonnegative on the closed interval $[0, 4]$, the area of the region is

$$\text{Area} = \int_0^4 (4x - x^2)\,dx.$$

A straightforward technique for evaluating a definite integral such as this will be discussed in Section 4.4. For now, however, you can evaluate a definite integral in two ways—you can use the limit definition *or* you can check to see whether the definite integral represents the area of a common geometric region, such as a rectangle, triangle, or semicircle.

EXAMPLE 3 Areas of Common Geometric Figures

Sketch the region corresponding to each definite integral. Then evaluate each integral using a geometric formula.

a. $\int_1^3 4\,dx$ **b.** $\int_0^3 (x + 2)\,dx$ **c.** $\int_{-2}^2 \sqrt{4 - x^2}\,dx$

Solution A sketch of each region is shown in Figure 4.24.

a. This region is a rectangle of height 4 and width 2.

$$\int_1^3 4\,dx = (\text{Area of rectangle}) = 4(2) = 8$$

b. This region is a trapezoid with an altitude of 3 and parallel bases of lengths 2 and 5. The formula for the area of a trapezoid is $\frac{1}{2}h(b_1 + b_2)$.

$$\int_0^3 (x + 2)\,dx = (\text{Area of trapezoid}) = \frac{1}{2}(3)(2 + 5) = \frac{21}{2}$$

c. This region is a semicircle of radius 2. The formula for the area of a semicircle is $\frac{1}{2}\pi r^2$.

$$\int_{-2}^2 \sqrt{4 - x^2}\,dx = (\text{Area of semicircle}) = \frac{1}{2}\pi(2^2) = 2\pi$$

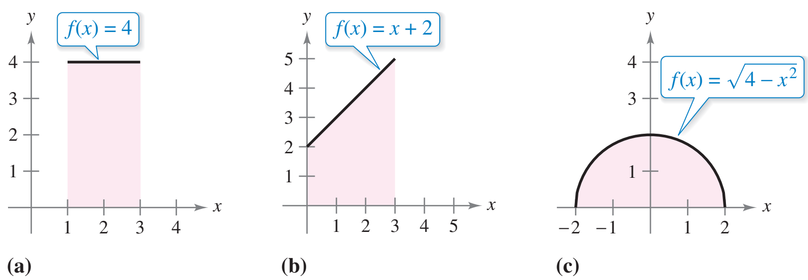


Figure 4.24

The variable of integration in a definite integral is sometimes called a *dummy variable* because it can be replaced by any other variable without changing the value of the integral. For instance, the definite integrals

$$\int_0^3 (x + 2)\,dx \quad \text{and} \quad \int_0^3 (t + 2)\,dt$$

have the same value.

Properties of Definite Integrals

The definition of the definite integral of f on the interval $[a, b]$ specifies that $a < b$. Now, however, it is convenient to extend the definition to cover cases in which $a = b$ or $a > b$. Geometrically, the next two definitions seem reasonable. For instance, it makes sense to define the area of a region of zero width and finite height to be 0.

Definitions of Two Special Definite Integrals

1. If f is defined at $x = a$, then $\int_a^a f(x)\,dx = 0$.
2. If f is integrable on $[a, b]$, then $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$.

EXAMPLE 4 Evaluating Definite Integrals

See LarsonCalculus.com for an interactive version of this type of example.

Evaluate each definite integral.

a. $\int_\pi^\pi \sin x\,dx$ **b.** $\int_3^0 (x + 2)\,dx$

Solution

a. Because the sine function is defined at $x = \pi$, and the upper and lower limits of integration are equal, you can write

$$\int_\pi^\pi \sin x\,dx = 0.$$

b. The integral $\int_3^0 (x + 2)\,dx$ is the same as that given in Example 3(b) except that the upper and lower limits are interchanged. Because the integral in Example 3(b) has a value of $\frac{21}{2}$, you can write

$$\int_3^0 (x + 2)\,dx = -\int_0^3 (x + 2)\,dx = -\frac{21}{2}.$$

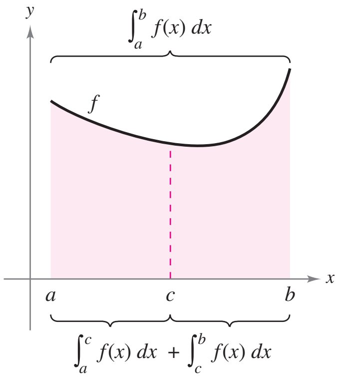


Figure 4.25

In Figure 4.25, the larger region can be divided at $x = c$ into two subregions whose intersection is a line segment. Because the line segment has zero area, it follows that the area of the larger region is equal to the sum of the areas of the two smaller regions.

THEOREM 4.6 Additive Interval Property

If f is integrable on the three closed intervals determined by a, b, and c, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

EXAMPLE 5 Using the Additive Interval Property

$$\int_{-1}^1 |x|\,dx = \int_{-1}^0 -x\,dx + \int_0^1 x\,dx \quad \text{Theorem 4.6}$$

$$= \frac{1}{2} + \frac{1}{2} \quad \text{Area of a triangle}$$

$$= 1$$

Because the definite integral is defined as the limit of a sum, it inherits the properties of summation given at the top of page 255.

THEOREM 4.7 Properties of Definite Integrals

If f and g are integrable on $[a, b]$ and k is a constant, then the functions kf and $f \pm g$ are integrable on $[a, b]$, and

1. $\displaystyle\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$

2. $\displaystyle\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx.$

REMARK Property 2 of Theorem 4.7 can be extended to cover any finite number of functions (see Example 6).

EXAMPLE 6 Evaluation of a Definite Integral

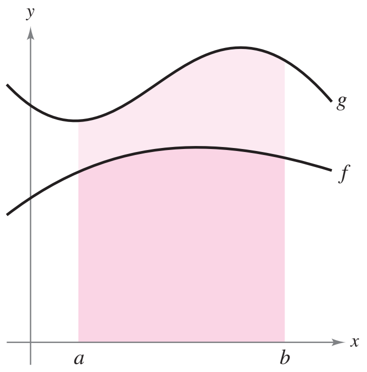
Evaluate $\displaystyle\int_1^3 (-x^2 + 4x - 3)\,dx$ using each of the following values.

$$\int_1^3 x^2\,dx = \frac{26}{3}, \qquad \int_1^3 x\,dx = 4, \qquad \int_1^3 dx = 2$$

Solution

$$\begin{aligned}\int_1^3 (-x^2 + 4x - 3)\,dx &= \int_1^3 (-x^2)\,dx + \int_1^3 4x\,dx + \int_1^3 (-3)\,dx \\ &= -\int_1^3 x^2\,dx + 4\int_1^3 x\,dx - 3\int_1^3 dx \\ &= -\left(\frac{26}{3}\right) + 4(4) - 3(2) \\ &= \frac{4}{3}\end{aligned}$$

If f and g are continuous on the closed interval $[a, b]$ and $0 \le f(x) \le g(x)$ for $a \le x \le b$, then the following properties are true. First, the area of the region bounded by the graph of f and the x-axis (between a and b) must be nonnegative. Second, this area must be less than or equal to the area of the region bounded by the graph of g and the x-axis (between a and b), as shown in Figure 4.26. These two properties are generalized in Theorem 4.8.



$\displaystyle\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$

Figure 4.26

THEOREM 4.8 Preservation of Inequality

1. If f is integrable and nonnegative on the closed interval $[a, b]$, then

$$0 \le \int_a^b f(x)\,dx.$$

2. If f and g are integrable on the closed interval $[a, b]$ and $f(x) \le g(x)$ for every x in $[a, b]$, then

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx.$$

A proof of this theorem is given in Appendix A.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

4.3 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Evaluating a Limit In Exercises 1 and 2, use Example 1 as a model to evaluate the limit

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta x_i$$

over the region bounded by the graphs of the equations.

1. $f(x) = \sqrt{x}, \quad y = 0, \quad x = 0, \quad x = 3$

   $\left(\textit{Hint: } \text{Let } c_i = \dfrac{3i^2}{n^2}.\right)$

2. $f(x) = \sqrt[3]{x}, \quad y = 0, \quad x = 0, \quad x = 1$

   $\left(\textit{Hint: } \text{Let } c_i = \dfrac{i^3}{n^3}.\right)$

Evaluating a Definite Integral as a Limit In Exercises 3–8, evaluate the definite integral by the limit definition.

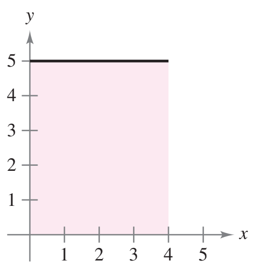
3. $\displaystyle\int_2^6 8\,dx$
4. $\displaystyle\int_{-2}^3 x\,dx$
5. $\displaystyle\int_{-1}^1 x^3\,dx$
6. $\displaystyle\int_1^4 4x^2\,dx$
7. $\displaystyle\int_1^2 (x^2 + 1)\,dx$
8. $\displaystyle\int_{-2}^1 (2x^2 + 3)\,dx$

Writing a Limit as a Definite Integral In Exercises 9–12, write the limit as a definite integral on the interval $[a, b]$, where c_i is any point in the ith subinterval.

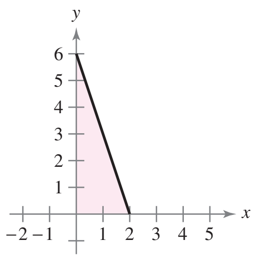
	Limit	Interval
9.	$\displaystyle\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} (3c_i + 10)\,\Delta x_i$	$[-1, 5]$
10.	$\displaystyle\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} 6c_i(4 - c_i)^2\,\Delta x_i$	$[0, 4]$
11.	$\displaystyle\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} \sqrt{c_i^2 + 4}\,\Delta x_i$	$[0, 3]$
12.	$\displaystyle\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} \left(\frac{3}{c_i^2}\right)\Delta x_i$	$[1, 3]$

Writing a Definite Integral In Exercises 13–22, set up a definite integral that yields the area of the region. (Do not evaluate the integral.)

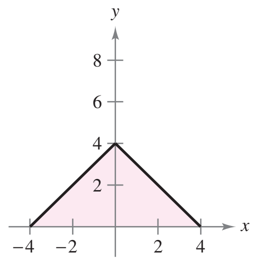
13. $f(x) = 5$



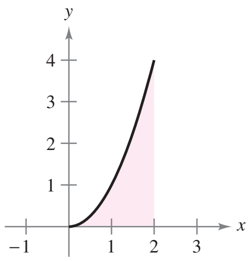
14. $f(x) = 6 - 3x$



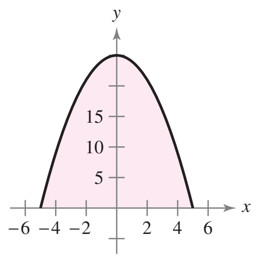
15. $f(x) = 4 - |x|$



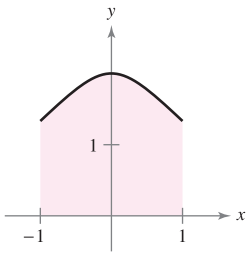
16. $f(x) = x^2$



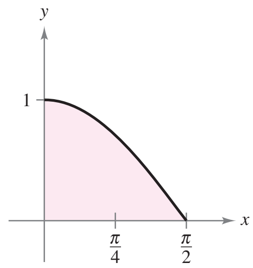
17. $f(x) = 25 - x^2$



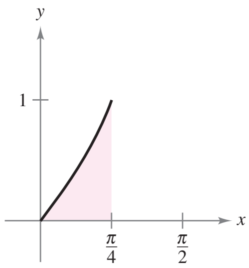
18. $f(x) = \dfrac{4}{x^2 + 2}$



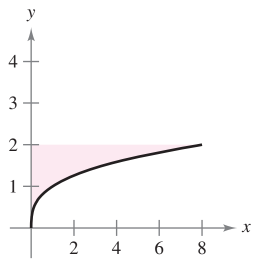
19. $f(x) = \cos x$



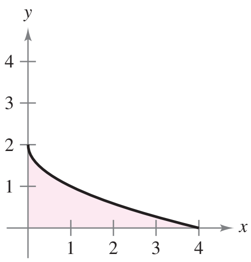
20. $f(x) = \tan x$



21. $g(y) = y^3$



22. $f(y) = (y - 2)^2$



Evaluating a Definite Integral Using a Geometric Formula In Exercises 23–32, sketch the region whose area is given by the definite integral. Then use a geometric formula to evaluate the integral $(a > 0, r > 0)$.

23. $\displaystyle\int_0^3 4\,dx$
24. $\displaystyle\int_{-4}^6 6\,dx$
25. $\displaystyle\int_0^4 x\,dx$
26. $\displaystyle\int_0^8 \frac{x}{4}\,dx$

27. $\int_0^2 (3x + 4)\, dx$

28. $\int_0^3 (8 - 2x)\, dx$

29. $\int_{-1}^1 (1 - |x|)\, dx$

30. $\int_{-a}^a (a - |x|)\, dx$

31. $\int_{-7}^7 \sqrt{49 - x^2}\, dx$

32. $\int_{-r}^r \sqrt{r^2 - x^2}\, dx$

Using Properties of Definite Integrals In Exercises 33–40, evaluate the integral using the following values.

$$\int_2^4 x^3\, dx = 60, \qquad \int_2^4 x\, dx = 6, \qquad \int_2^4 dx = 2$$

33. $\int_4^2 x\, dx$

34. $\int_2^2 x^3\, dx$

35. $\int_2^4 8x\, dx$

36. $\int_2^4 25\, dx$

37. $\int_2^4 (x - 9)\, dx$

38. $\int_2^4 (x^3 + 4)\, dx$

39. $\int_2^4 \left(\frac{1}{2}x^3 - 3x + 2\right) dx$

40. $\int_2^4 (10 + 4x - 3x^3)\, dx$

41. Using Properties of Definite Integrals Given

$$\int_0^5 f(x)\, dx = 10 \quad \text{and} \quad \int_5^7 f(x)\, dx = 3$$

evaluate

(a) $\int_0^7 f(x)\, dx.$ (b) $\int_5^0 f(x)\, dx.$

(c) $\int_5^5 f(x)\, dx.$ (d) $\int_0^5 3f(x)\, dx.$

42. Using Properties of Definite Integrals Given

$$\int_0^3 f(x)\, dx = 4 \quad \text{and} \quad \int_3^6 f(x)\, dx = -1$$

evaluate

(a) $\int_0^6 f(x)\, dx.$ (b) $\int_6^3 f(x)\, dx.$

(c) $\int_3^3 f(x)\, dx.$ (d) $\int_3^6 -5f(x)\, dx.$

43. Using Properties of Definite Integrals Given

$$\int_2^6 f(x)\, dx = 10 \quad \text{and} \quad \int_2^6 g(x)\, dx = -2$$

evaluate

(a) $\int_2^6 [f(x) + g(x)]\, dx.$ (b) $\int_2^6 [g(x) - f(x)]\, dx.$

(c) $\int_2^6 2g(x)\, dx.$ (d) $\int_2^6 3f(x)\, dx.$

44. Using Properties of Definite Integrals Given

$$\int_{-1}^1 f(x)\, dx = 0 \quad \text{and} \quad \int_0^1 f(x)\, dx = 5$$

evaluate

(a) $\int_{-1}^0 f(x)\, dx.$ (b) $\int_0^1 f(x)\, dx - \int_{-1}^0 f(x)\, dx.$

(c) $\int_{-1}^1 3f(x)\, dx.$ (d) $\int_0^1 3f(x)\, dx.$

45. Estimating a Definite Integral Use the table of values to find lower and upper estimates of

$$\int_0^{10} f(x)\, dx.$$

Assume that f is a decreasing function.

x	0	2	4	6	8	10
$f(x)$	32	24	12	−4	−20	−36

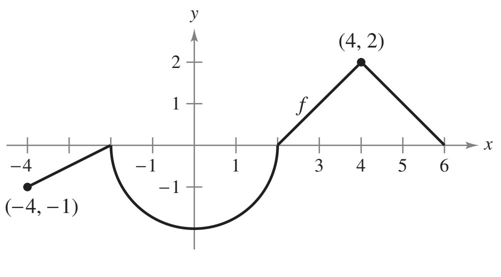
46. Estimating a Definite Integral Use the table of values to estimate

$$\int_0^6 f(x)\, dx.$$

Use three equal subintervals and the (a) left endpoints, (b) right endpoints, and (c) midpoints. When f is an increasing function, how does each estimate compare with the actual value? Explain your reasoning.

x	0	1	2	3	4	5	6
$f(x)$	−6	0	8	18	30	50	80

47. Think About It The graph of f consists of line segments and a semicircle, as shown in the figure. Evaluate each definite integral by using geometric formulas.

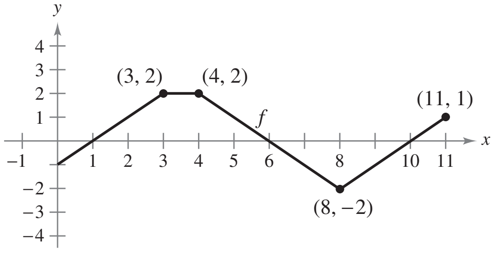


(a) $\int_0^2 f(x)\, dx$ (b) $\int_2^6 f(x)\, dx$

(c) $\int_{-4}^2 f(x)\, dx$ (d) $\int_{-4}^6 f(x)\, dx$

(e) $\int_{-4}^6 |f(x)|\, dx$ (f) $\int_{-4}^6 [f(x) + 2]\, dx$

48. **Think About It** The graph of f consists of line segments, as shown in the figure. Evaluate each definite integral by using geometric formulas.



(a) $\int_0^1 -f(x)\,dx$ (b) $\int_3^4 3f(x)\,dx$

(c) $\int_0^7 f(x)\,dx$ (d) $\int_5^{11} f(x)\,dx$

(e) $\int_0^{11} f(x)\,dx$ (f) $\int_4^{10} f(x)\,dx$

49. **Think About It** Consider the function f that is continuous on the interval $[-5, 5]$ and for which

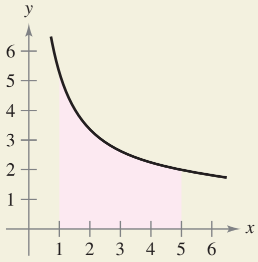
$$\int_0^5 f(x)\,dx = 4.$$

Evaluate each integral.

(a) $\int_0^5 [f(x) + 2]\,dx$ (b) $\int_{-2}^3 f(x + 2)\,dx$

(c) $\int_{-5}^5 f(x)\,dx$ (f is even.) (d) $\int_{-5}^5 f(x)\,dx$ (f is odd.)

50. **HOW DO YOU SEE IT?** Use the figure to fill in the blank with the symbol <, >, or =. Explain your reasoning.



(a) The interval $[1, 5]$ is partitioned into n subintervals of equal width Δx, and x_i is the left endpoint of the ith subinterval.

$$\sum_{i=1}^{n} f(x_i)\,\Delta x \quad \square \quad \int_1^5 f(x)\,dx$$

(b) The interval $[1, 5]$ is partitioned into n subintervals of equal width Δx, and x_i is the right endpoint of the ith subinterval.

$$\sum_{i=1}^{n} f(x_i)\,\Delta x \quad \square \quad \int_1^5 f(x)\,dx$$

51. **Think About It** A function f is defined below. Use geometric formulas to find $\int_0^8 f(x)\,dx$.

$$f(x) = \begin{cases} 4, & x < 4 \\ x, & x \geq 4 \end{cases}$$

52. **Think About It** A function f is defined below. Use geometric formulas to find $\int_0^{12} f(x)\,dx$.

$$f(x) = \begin{cases} 6, & x > 6 \\ -\frac{1}{2}x + 9, & x \leq 6 \end{cases}$$

WRITING ABOUT CONCEPTS

Approximation **In Exercises 53–56, determine which value best approximates the definite integral. Make your selection on the basis of a sketch.**

53. $\int_0^4 \sqrt{x}\,dx$

(a) 5 (b) −3 (c) 10 (d) 2 (e) 8

54. $\int_0^{1/2} 4\cos \pi x\,dx$

(a) 4 (b) $\frac{4}{3}$ (c) 16 (d) 2π (e) −6

55. $\int_0^1 2\sin \pi x\,dx$

(a) 6 (b) $\frac{1}{2}$ (c) 4 (d) $\frac{5}{4}$

56. $\int_0^9 (1 + \sqrt{x})\,dx$

(a) −3 (b) 9 (c) 27 (d) 3

57. **Determining Integrability** Determine whether the function

$$f(x) = \frac{1}{x - 4}$$

is integrable on the interval $[3, 5]$. Explain.

58. **Finding a Function** Give an example of a function that is integrable on the interval $[-1, 1]$, but not continuous on $[-1, 1]$.

Finding Values **In Exercises 59–62, find possible values of a and b that make the statement true. If possible, use a graph to support your answer. (There may be more than one correct answer.)**

59. $\int_{-2}^1 f(x)\,dx + \int_1^5 f(x)\,dx = \int_a^b f(x)\,dx$

60. $\int_{-3}^3 f(x)\,dx + \int_3^6 f(x)\,dx - \int_a^b f(x)\,dx = \int_{-1}^6 f(x)\,dx$

61. $\int_a^b \sin x\,dx < 0$

62. $\int_a^b \cos x\,dx = 0$

True or False? **In Exercises 63–68, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

63. $\displaystyle\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

64. $\displaystyle\int_a^b f(x)g(x)\,dx = \left[\int_a^b f(x)\,dx\right]\left[\int_a^b g(x)\,dx\right]$

65. If the norm of a partition approaches zero, then the number of subintervals approaches infinity.

66. If f is increasing on $[a, b]$, then the minimum value of $f(x)$ on $[a, b]$ is $f(a)$.

67. The value of

$$\int_a^b f(x)\,dx$$

must be positive.

68. The value of

$$\int_2^2 \sin(x^2)\,dx$$

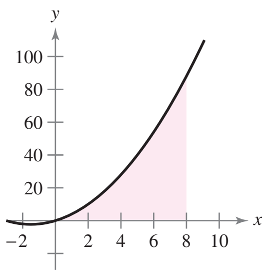
is 0.

69. Finding a Riemann Sum Find the Riemann sum for $f(x) = x^2 + 3x$ over the interval $[0, 8]$, where

$$x_0 = 0,\quad x_1 = 1,\quad x_2 = 3,\quad x_3 = 7,\quad \text{and}\quad x_4 = 8$$

and where

$$c_1 = 1,\quad c_2 = 2,\quad c_3 = 5,\quad \text{and}\quad c_4 = 8.$$

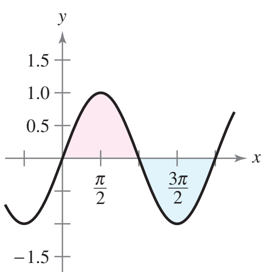


70. Finding a Riemann Sum Find the Riemann sum for $f(x) = \sin x$ over the interval $[0, 2\pi]$, where

$$x_0 = 0,\quad x_1 = \frac{\pi}{4},\quad x_2 = \frac{\pi}{3},\quad x_3 = \pi,\quad \text{and}\quad x_4 = 2\pi,$$

and where

$$c_1 = \frac{\pi}{6},\quad c_2 = \frac{\pi}{3},\quad c_3 = \frac{2\pi}{3},\quad \text{and}\quad c_4 = \frac{3\pi}{2}.$$



71. Proof Prove that $\displaystyle\int_a^b x\,dx = \frac{b^2 - a^2}{2}$.

72. Proof Prove that $\displaystyle\int_a^b x^2\,dx = \frac{b^3 - a^3}{3}$.

73. Think About It Determine whether the Dirichlet function

$$f(x) = \begin{cases} 1, & x \text{ is rational} \\ 0, & x \text{ is irrational} \end{cases}$$

is integrable on the interval $[0, 1]$. Explain.

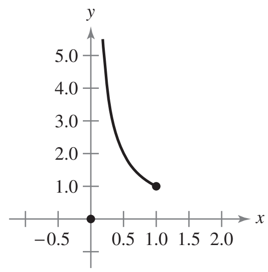
74. Finding a Definite Integral The function

$$f(x) = \begin{cases} 0, & x = 0 \\ \dfrac{1}{x}, & 0 < x \le 1 \end{cases}$$

is defined on $[0, 1]$, as shown in the figure. Show that

$$\int_0^1 f(x)\,dx$$

does not exist. Why doesn't this contradict Theorem 4.4?



75. Finding Values Find the constants a and b that maximize the value of

$$\int_a^b (1 - x^2)\,dx.$$

Explain your reasoning.

76. Step Function Evaluate, if possible, the integral

$$\int_0^2 [\![x]\!]\,dx.$$

77. Using a Riemann Sum Determine

$$\lim_{n\to\infty} \frac{1}{n^3}[1^2 + 2^2 + 3^2 + \cdot\cdot\cdot + n^2]$$

by using an appropriate Riemann sum.

PUTNAM EXAM CHALLENGE

78. For each continuous function $f: [0, 1] \to \mathbb{R}$, let

$$I(f) = \int_0^1 x^2 f(x)\,dx \quad \text{and} \quad J(x) = \int_0^1 x(f(x))^2\,dx.$$

Find the maximum value of $I(f) - J(f)$ over all such functions f.

This problem was composed by the Committee on the Putnam Prize Competition.